Mon-Fri MATH1510

Week 6

Topics Covered:

- Equation of Tangent Line
- Taylor's Theorem/Equation of Tg. Line/Differential
- Start of Proof of Taylor's Theorem

Given a function, differentiable at a point $x = c$ in its domain, we can then talk about the "tangent line" at $x = c$. (or simply "at c"). How to write down the equation of this straight line (which passes through the point $(c, f(c))$)?

There are many ways to do it. We do it via the Taylor's Theorem, which says

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \text{Error term}$$

--(1)

The LHS (= left-hand side) of this equation is an expression describing the curve

$$y = f(x).$$

The RHS (= right-hand side) is a "polynomial-like expression" in $(x-c)^{power}$ given by

$$y = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

if we "stop" at the n^{th} term. (Note that: $f(c), \frac{f'(c)}{1!}, \frac{f''(c)}{2!}, \cdots, \frac{f^{(n)}(c)}{n!}$ are numbers!)

Examples:

- (Stopping at the 0th term) Then equation (1) above gives

$$f(x) = f(c) + E_0(x, c).$$

- (Stopping at the 1st term) Then equation (1) above gives

$$f(x) = f(c) + f'(c)(x-c) + E_1(x, c)$$

  Here the LHS is about the curve: $y = f(x)$. The RHS is about the "tangent line" $y = f(c) + f'(c)(x-c)$ together with an "error term".

We can approximate, e.g. $\sin(0.5)$ using this kind of "linearization" method at a chosen point, say $c = 0$.

If we stop at the 2nd term, we can see whether the function is "concave upward" or "downward". (This is useful when we discuss "relative (or "local") maximum/minimum points).

Relation to "Differential", i.e. $\boldsymbol{dy, dx}$

(read also the article in the link: http://mathinsight.org/linear_approximation_differentials_refresher)

In many textbooks, you will find the word "differential". Now we try to explain this (by relating it to the equation of the tangent line).

Remember that, we have shown the following:

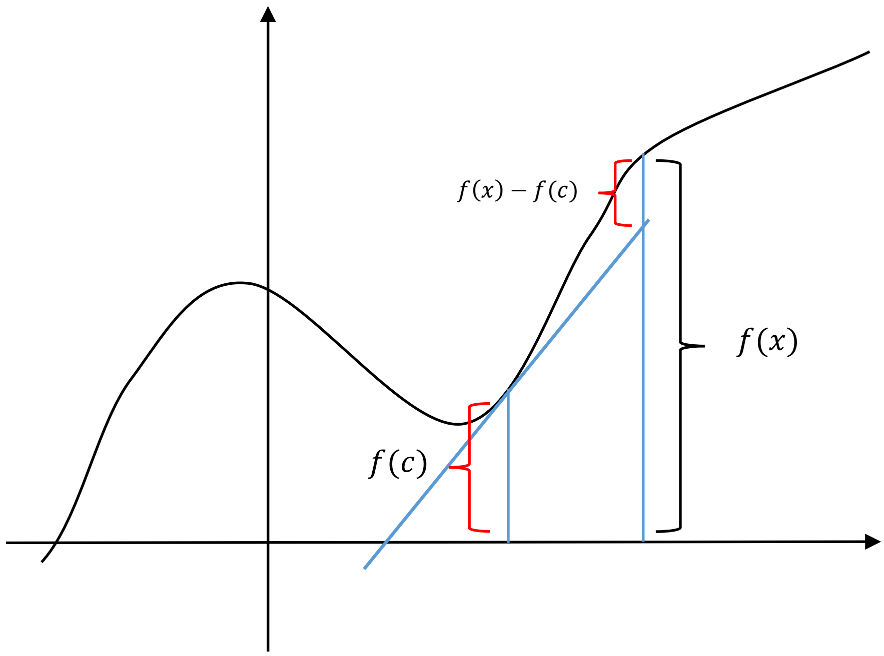
$$f(x) = f(c) + f'(c)(x - c) + E_1(x, c)$$

Rewriting it gives

$$f(x) - f(c) = f'(c)(x - c) + E_1(x, c) \text{ --------------------------------(2)}$$

Now, the LHS is a "difference between the values of $f(x)$ and $f(c)$" (i.e. the value of the function at x and at c).

See the picture below:



The RHS is "slope (i.e. $f'(x)$)" of the right-angled triangle multiplied by the base length (i.e. $x - c$).

If we let $\Delta y = f(x) - f(c)$ (Δy means "difference in y –coordinate") and $\Delta x =$

$x - c$ ("difference in x —coordinate), then equation (2) takes the form

$$\Delta y = f'(c)\Delta x + E_1(x, c)$$

or (if we throw away the "error"term):

$$\Delta y \approx f'(c)\Delta x \text{ ------------------------------ (3)}$$

Until now, everything is just equation (1). Now we define two new notations:

1. We give a symbol to the expression (recall that $f'(x) = \frac{dy}{dx}$)

$$f'(c)dx$$

   and denote it as dy. (So dy is just another way of saying "$f'(c)dx$")
2. Using this notation (i.e. $dy = f'(c)dx$), equation (3) becomes

$$\Delta y \approx dy = f'(c)\Delta x \text{ --------------(4)}$$

3. (***) In the following, we try to explain why one can write $\Delta x = dx$. The reason is as follows:
   - We have defined $dy = f'(x)\Delta x$ (slightly different from "$f'(c)\Delta x$"!)
   - Now $f'(x) = \frac{df}{dx}$ or $\frac{dy}{dx}$, so $dy = \left(\frac{dy}{dx}\right)\Delta x$. This formula means the yfollowing: "Differential of y = derivative of y $\times \Delta x$"
     If now our function is just x. I.e. $y = x$, the this formula gives "Differential of x = derivative of x $\times$ Δx."

     Or "$dx = \left(\frac{dx}{dx}\right)\Delta x$ which means $dx = \Delta x$."

Conclusion:

Combining (4), i.e.

$$\Delta y \approx f'(c)\Delta x$$

and $\Delta x = \left(\frac{dx}{dx}\right)dx$ we get the "approximate" formula

$$\Delta y \approx f'(c)dx$$

If you like, you can write also $\Delta y \approx \frac{dy}{dx}|_{x=c}dx$

Remark:

In many textbooks, you may find instead of $\Delta y \approx f'(c)dx$ the writing $\Delta y \approx f'(x)dx$ (this way of writing down the differential of y is an "abuse" of notations).

Start of Proof of T.T.

(Idea) First think simple. The 0^{th} order case of T.T. is:

$$f(x) = f(c) + E_0(x, c)$$

which can be rewritten in the form

$$f(x) - f(c) = E_0(x, c)$$

Now dividing through by $x - c$ gives

$$\frac{f(x) - f(c)}{x - c} = \frac{E_0(x, c)}{x - c}$$

The left-hand side is equal to $f'(\xi)$ on Monday.